TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 270, Number 2, April 1982

THE NONFINITE GENERATION OF Aut(G), G FREE METABELIAN OF RANK 3

BY

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ABSTRACT. The group of automorphisms of the free metabelian group of rank 3 is not finitely generated.

Let H be a free solvable group of rank $n \geqslant 2$, $\operatorname{Aut}(H)$ the automorphism group of H, and $\operatorname{Inn}(H)$ the group of inner automorphisms of H. For $n = 2$, it was shown by the authors and E. Formanek [**4**, Theorem 1] that $\operatorname{Aut}(H)/\operatorname{Inn}(H) \cong \mathrm{GL}_2(\mathbf{Z})$, $\mathbf{Z}$ the ring of integers. One consequence of this fact is that $\operatorname{Aut}(H)$ is finitely generated (f.g.). The purpose of this paper is to prove the following contrasting theorem.

THEOREM. *If G is the free metabelian group of rank three, then* $\operatorname{Aut}(G)$ *is not finitely generated.*

This result may also be compared with a theorem of L. Auslander [**1**], which states that the automorphism group of a polycyclic group is finitely presented.

If G' denotes the commutator subgroup of G, then the kernel of the natural homomorphism $\operatorname{Aut}(G) \to \operatorname{Aut}(G/G')$ is called the group of IA-automorphisms of G and will be denoted by $\mathrm{IA}(G)$. $\mathrm{IA}(G)$, which contains $\operatorname{Inn}(G)$, was shown by the authors [**3**] to be non-f.g. for G as in the Theorem. To prove that $\operatorname{Aut}(G)$ is not f.g., we need to show that $\mathrm{IA}(G)$ is not f.g. as an $\operatorname{Aut}(G/G') \cong \mathrm{GL}_3(\mathbf{Z})$-operator group.

The question arises concerning $\operatorname{Aut}(H)$ where H is free metabelian of rank $n > 3$. The immediate feeling that $\operatorname{Aut}(H)$ is also non-f.g. may be incorrect. For definiteness let $n = 4$. Suppose G is free metabelian of rank 3, and consider $\operatorname{Aut}(G)$ as embedded in $\operatorname{Aut}(H)$ in an obvious manner. The nonfinite generatiion of $\operatorname{Aut}(G)$ comes from the existence of "nontame" automorphisms in $\operatorname{Aut}(G)$, i.e., automorphisms of $G \cong F/F''$ which are not induced by automorphisms of the free group F of rank three, and the necessity to include infinitely many nontame automorphisms in any generating set for $\operatorname{Aut}(G)$. However, the authors have discovered that many of the nontame automorphisms become tame when considered as elements in $\operatorname{Aut}(H)$. Whether this phenomenon is true for all nontame elements of $\operatorname{Aut}(G)$ is unknown as yet, but the question of the finite or nonfinite generation of $\operatorname{Aut}(H)$ must be considered a difficult open problem, with the interesting possibility of finite generation as the answer.

Received by the editors December 31, 1980.

1980 *Mathematics Subject Classification*. Primary 20F28.

Key words and phrases. Free solvable group, linear group, free product with amalgamation.

[1]Research of both authors was supported by NSF Grant MCS-7901738.

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0002-9947/81/0000-1017/$03.00

Throughout the rest of this paper G is the free metabelian group of rank 3 and $A = G/G'$ the free abelian group of rank 3.

$\mathbf{Z}A$ will denote the integral group ring of A, S and T multiplicatively closed sets in $\mathbf{Z}A$ not containing the zero element, and $\mathbf{Z}A_S$ ($\mathbf{Z}A_T$) the localization of $\mathbf{Z}A$ with respect to S (respectively T). Since we shall be working almost exclusively with elements from A and $\operatorname{Aut}(A)$ rather than with elements from G and $\operatorname{Aut}(G)$, we adopt the convention that elements from G and $\operatorname{Aut}(G)$ will be starred while elements from A and $\operatorname{Aut}(A)$ will not be starred. If $g^* \in G$, g will denote its image in A, and if $\phi^* \in \operatorname{Aut}(G)$, ϕ will denote its image in $\operatorname{Aut}(A)$.

1. A faithful representation of $\operatorname{Aut}(G)$. Let $\phi \in \operatorname{Aut}(A)$. Then ϕ extends in the usual way to an automorphism of $\mathbf{Z}A$, also denoted by ϕ. If $(a_{ij}) \in \mathrm{GL}_3(\mathbf{Z}A)$, then $(a_{ij}) \to (a_{ij}^{\phi})$ defines an automorphism of $\mathrm{GL}_3(\mathbf{Z}A)$. Let $\mathcal{Q}$ denote the semidirect product of $\operatorname{Aut}(A)$ and $\mathrm{GL}_3(\mathbf{Z}A)$ under this action. Thus, $\mathcal{Q}$ is the group consisting of all pairs $[\phi, (a_{ij})]$ in $\operatorname{Aut}(A) \times \mathrm{GL}_3(\mathbf{Z}A)$ with multiplication defined by

$$[\phi, (a_{ij})][\psi, (b_{ij})] = [\phi\psi, (a_{ij}^{\psi})(b_{ij})].$$

We now describe the Magnus representation of G and the induced faithful representation of $\operatorname{Aut}(G)$.

LEMMA 1. (i) (*Magnus* [7]) *Let t_1, t_2, t_3 be a basis for a free left $\mathbf{Z}A$-module, and x, y, z free generators of A. Then, the matrices*

$$x^* = \begin{pmatrix} x & t_1 \\ 0 & 1 \end{pmatrix}, \qquad y^* = \begin{pmatrix} y & t_2 \\ 0 & 1 \end{pmatrix}, \qquad z^* = \begin{pmatrix} z & t_3 \\ 0 & 1 \end{pmatrix}$$

are free generators of a free metabelian group.

(ii) (*Bachmuth* [**2**, *Lemma* 1], *Remeslennikov and Sokolov* [**8**, *Theorem* 2]) *The matrix*

$$\begin{pmatrix} g & a_1t_1 + a_2t_2 + a_3t_3 \\ 0 & 1 \end{pmatrix}, \qquad g \in A, \quad a_i \in \mathbf{Z}A,$$

is contained in the group generated by x^*, y^*, z^* if and only if

$$(1 - g) = a_1(1 - x) + a_2(1 - y) + a_3(1 - z). \quad \blacksquare$$

From now on, we identify G with the group generated by x^*, y^*, z^* in Lemma 1. Suppose that $\phi^* \in \operatorname{Aut}(G)$ induces $\phi \in \operatorname{Aut}(A)$. ϕ^* is uniquely determined by its action on x^*, y^*, z^*.

$$\phi^*: \quad \begin{aligned} x^* &\to \begin{pmatrix} x^{\phi} & a_{11}t_1 + a_{12}t_2 + a_{13}t_3 \\ 0 & 1 \end{pmatrix}, \\ y^* &\to \begin{pmatrix} y^{\phi} & a_{21}t_1 + a_{22}t_2 + a_{23}t_3 \\ 0 & 1 \end{pmatrix}, \\ z^* &\to \begin{pmatrix} z^{\phi} & a_{31}t_1 + a_{32}t_2 + a_{33}t_3 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus, we have an embedding of Aut(G) into $\mathcal{Q}$ given by $\phi^* \to [\phi, (a_{ij})]$, where

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1-x \\ 1-y \\ 1-z \end{pmatrix} = \begin{pmatrix} 1-x^{\phi} \\ 1-y^{\phi} \\ 1-z^{\phi} \end{pmatrix}.$$

Let S be the multiplicative monoid of $\mathbf{Z}A$ generated by all $(1-x^{\phi})$, $\phi \in \operatorname{Aut}(A)$. ϕ extends to an automorphism (also denoted by ϕ) of $\mathbf{Z}A_S$. Hence, it is possible to define the semidirect product $\mathcal{B}$ of Aut(A) and $\mathrm{GL}_2(\mathbf{Z}A_S)$ under the action of Aut(A) on $\mathrm{GL}_2(\mathbf{Z}A_S)$.

LEMMA 2. *The following map defines a homomorphism of* Aut(G) *into* $\mathcal{B}$: *First embed* Aut(G) *into* $\mathcal{Q}$ *via* $\phi^* \to [\phi, (a_{ij})]$ *as above, and then map* $[\phi, (a_{ij})]$ *into* $[\phi, (b_{ij})]$ *where* (b_{ij}) *is the* 2×2 *matrix*

$$\begin{pmatrix} [a_{22}(1-x^{\phi}) - a_{12}(1-y^{\phi})](1-x)^{-1} & a_{23}(1-x^{\phi}) - a_{13}(1-y^{\phi}) \\ [a_{32}(1-x^{\phi}) - a_{12}(1-z^{\phi})](1-x^{\phi})^{-1}(1-x)^{-1} & [a_{33}(1-x^{\phi}) - a_{13}(1-z^{\phi})](1-x^{\phi})^{-1} \end{pmatrix}.$$

PROOF. After identifying Aut(G) with its embedding in $\mathcal{Q}$, we conjugate Aut(G) by the element

$$[1, (c_{ij})] = \left[1, \begin{pmatrix} (1-x) & 0 & 0 \\ (1-y) & (1-x)^{-1} & 0 \\ (1-z) & 0 & 1 \end{pmatrix}\right].$$

A routine computation verifies that

$$[1,(c_{ij})]^{-1}[\phi,(a_{ij})][1,(c_{ij})]$$
$$= \left[\phi, \begin{pmatrix} 1 & a_{12}(1-x)^{-1}(1-x^{\phi})^{-1} & a_{13}(1-x^{\phi})^{-1} \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix}\right]. \quad \blacksquare$$

Next we formulate Ihara's Theorem. Since $\mathbf{Z}A = \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$, the quotient field $Q = Q(\mathbf{Z}A)$ of $\mathbf{Z}A$ has a discrete valuation determined by the powers of z. To be more specific, if $0 \neq \alpha \in \mathbf{Z}A$, then α can be uniquely written as $\alpha = \sum_{i=m}^{n} a_i z^i = z^m \sum_{i=0}^{n-m} a_{i+m} z^i$, $m < n$. Define the z-value of α to be $v(\alpha) = m$. If $\alpha \neq 0$, $\beta \neq 0$ are in $\mathbf{Z}A$, then define $v(\alpha/\beta) = v(\alpha) - v(\beta)$.

Let $\mathcal{O}$ be the valuation ring in Q with respect to v.

LEMMA 3 [**10**, *p.* 110]. $\mathrm{SL}_2(Q)$ *is the free product of* $\mathrm{SL}_2(\mathcal{O})$ *and* $\mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)} = \left(\begin{smallmatrix}1&0\\0&z^{-1}\end{smallmatrix}\right)\mathrm{SL}_2(\mathcal{O})\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)$ *with their intersection* Γ *amalgamated.*

In symbols

$$\mathrm{SL}_2(Q) = \mathrm{SL}_2(\mathcal{O}) *_{\Gamma} \mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)}$$

where $\Gamma = \mathrm{SL}_2(\mathcal{O}) \cap \mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)}$. $\blacksquare$

In the final lemma of this section, we collect together several well-known results used in the next section.

LEMMA 4. (i) (*P. Hall* [**9**, *p.* 32]) *If N is a normal subgroup of a f.g. group H such that H/N is finitely presented (f.p.), then N is f.g. as a G-operator group.*

(ii) [**9**, p. 32] *The class of f.p. groups is closed under forming extensions.*

(iii) *The group of units of* $\mathbf{Z}A$ *is* $\pm A$ *and therefore is f.p.*

(iv) $\mathrm{GL}_3(\mathbf{Z})$ *is f.p.* ■

2. $\mathrm{Aut}(G)$ **is not finitely generated.** Let N be the normal subgroup of $\mathrm{Aut}(G)$ consisting of all $[1,(a_{ij})]$ such that $\det(a_{ij}) = 1$. Then, the sequences

$$1 \to \mathrm{IA}(G) \to \mathrm{Aut}(G) \to \mathrm{GL}_3(\mathbf{Z}) \to 1 \quad \text{and} \quad 1 \to N \to \mathrm{IA}(G) \to \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z} \to 1$$

are exact. Therefore, $\mathrm{IA}(G)/N$ and $\mathrm{Aut}(G)/\mathrm{IA}(G)$ are both f.p., and we conclude from Lemma 4(ii) that $\mathrm{Aut}(G)/N$ is f.p. Thus, if $\mathrm{Aut}(G)$ is f.g., then N is f.g. as an $\mathrm{Aut}(G)$-operator group.

Henceforth, in order to reach a contradiction, we assume that $\mathrm{Aut}(G)$ *is a f.g. group and N is a f.g.* $\mathrm{Aut}(G)$*-operator group.*

Let $\mathcal{C}$ and $\mathfrak{N}$ be the images of $\mathrm{Aut}(G)$ and N in $\mathfrak{B}$, the semidirect product of $\mathrm{Aut}(A)$ and $\mathrm{GL}_2(\mathbf{Z}A_S)$ under the representation described in §1. Let $[\phi_1,(a_{ij1})],\ldots,[\phi_r,(a_{ijr})]$ generate $\mathcal{C}$ as a group, and let $[1,(b_{ij1})],\ldots,[1,(b_{ijs})]$ generate $\mathfrak{N}$ as a $\mathcal{C}$-operator group.

LEMMA 5. *There exist prime integers $p_1,\ldots,p_\mu$ and irreducible (noninteger) elements $f_1,\ldots,f_\nu$ in $\mathbf{Z}A$ such that if T is the multiplicative monoid generated by $\pm A$, S, p_i $(i = 1,\ldots,\mu)$, and f_j^ϕ $(j = 1,\ldots,\nu$; $\phi \in \mathrm{Aut}(A))$, then $(b_{ij}) \in E_2(\mathbf{Z}A_T)$ for all $[1,(b_{ij})] \in \mathfrak{N}$.*

PROOF. From the statement of Lemma 2 we see that the a_{12k} and the b_{12k} are in $\mathbf{Z}A$. Let $f_1,\ldots,f_\nu$ be a complete list (up to unit factor in $\mathbf{Z}A$) of the irreducible noninteger polynomials in $\mathbf{Z}A$ which appear as a factor of one of the a_{12k} or the b_{12k}. Let $p_1,\ldots,p_\mu$ be a complete list of the prime integers which are a factor of one of the a_{12k} or the b_{12k} or a factor of an integer coefficient of one of the f_k. Let T be defined as in the lemma. Then, a typical element τ of T can be expressed as

$$\tau = \pm p_{i_1}^{a_1}\cdots p_{i_l}^{a_l} z^{b_1}(1-g_1)\cdots z^{b_m}(1-g_m) g z^{c_1} f_{j_1}^{\psi_1} z^{c_2} f_{j_2}^{\psi_2}\cdots z^{c_n} f_{j_n}^{\psi_n}$$

where $l \geqslant 0$, $m \geqslant 0$, $n \geqslant 0$; $p_{i_k} \in \{p_1,\ldots,p_\mu\}$; $a_i \geqslant 1$; $(1-g_1),\ldots,(1-g_m) \in S$; $g \in A$; $f_{j_i} \in \{f_1,\ldots,f_\mu\}$; $b_i, c_i \in \mathbf{Z}$; and $\psi_i \in \mathrm{Aut}(A)$. (The reason we have included A in the set of generators of T is so that we may multiply the f_i^ϕ by any element in A and still remain in T.)

Notice that $\tau^\phi \in T$ for all $\tau \in T$ and for all $\phi \in \mathrm{Aut}(A)$. We now claim that if $[\phi,(a_{ij})] \in \mathcal{C}$, then $(a_{ij}) \in GE_2(\mathbf{Z}A_T)$. Since a_{12k}^ϕ is in T, it is clear that $(a_{ijk}^\phi) \in GE_2(\mathbf{Z}A_T)$, $1 \leqslant k \leqslant r$, $\phi \in \mathrm{Aut}(A)$. Thus, if $[\psi,(a_{ij})] \in \mathcal{C}$, then (a_{ij}) is a product of the (a_{ijk}^ϕ), whence $(a_{ij}) \in GE_2(\mathbf{Z}A_T)$.

We next note that if (c_{ij}) is in $GE_2(\mathbf{Z}A_T)$, resp. $E_2(\mathbf{Z}A_T)$, then $(c_{ij}^\phi) \in GE_2(\mathbf{Z}A_T)$, resp. $E_2(\mathbf{Z}A_T)$. Also, $b_{12k} \in T$, whence $(b_{i2k}^\phi) \in E_2(\mathbf{Z}A_T)$, $1 \leqslant k \leqslant s$, $\phi \in \mathrm{Aut}(A)$. $\mathfrak{N}$ is generated as a group by the elements

$$\left[\phi^{-1},\left(a_{ij}^{\phi^{-1}}\right)\right]\left[1,(b_{ijk})\right]\left[\phi,(a_{ij})\right] = \left[1,(a_{ij})^{-1}\left(b_{ijk}^\phi\right)(a_{ij})\right]$$

where $1 \leq k \leq s$, $[\phi, (a_{ij})] \in \mathcal{C}$. Since $E_2(\mathbf{Z}A_T)$ is a normal subgroup of $GE_2(\mathbf{Z}A_T)$, our previous work shows that $(a_{ij})^{-1}(b_{ijk}^{\phi})(a_{ij}) \in E_2(\mathbf{Z}A_T)$. Thus, $(b_{ij}) \in E_2(\mathbf{Z}A_T)$ if $[1, (b_{ij})] \in \mathfrak{N}$. ■

From now on, we identify $[1, (b_{ij})] \in \mathfrak{N}$ with $(b_{ij}) \in E_2(\mathbf{Z}A_T)$, i.e. *we regard $\mathfrak{N}$ as a subgroup of $E_2(\mathbf{Z}A_T)$.*

A typical element of $R = \mathbf{Z}A_T \cap \mathcal{O}$ can be expressed as f/τ where $f \in \mathbf{Z}[x, x^{-1}, y, y^{-1}, z]$ and $\tau \in T_z$. Here T_z is the submonoid of T consisting of all elements of z-value zero, i.e.,

$$\tau = \pm p_{i_1}^{a_1} \cdots p_{i_l}^{a_l} z^{b_1}(1 - g_1) \cdots z^{b_m}(1 - g_m) g z^{c_1} f_{j_1}^{\psi_1} z^{c_2} f_{j_2}^{\psi_2} \cdots z^{c_n} f_{j_n}^{\psi_n}$$

where g, the $z^{b_i}(1 - g_i)$ and the $z^{c_i} f_{j_i}^{\psi_i}$ have z-value zero. Thus $R = \mathbf{Z}A_T \cap \mathcal{O} = \mathbf{Z}[x, x^{-1}, y, y^{-1}, z]_{T_z}$.

Let q be a noninvertible prime integer in $\mathbf{Z}A_T$. Let T' denote the image of T in $\mathbf{Z}_qA$ under the natural map $\mathbf{Z}A \to \mathbf{Z}_qA$ where $\mathbf{Z}_q = \mathbf{Z}/q\mathbf{Z}$, and if U is a subset or element of $\mathbf{Z}_A$, let U' denote the image of U in $\mathbf{Z}_qA_{T'}$ under the natural map $\mathbf{Z}A_T \to \mathbf{Z}_qA_{T'}$. This convention also applies to subsets or elements of the quotient field $Q(\mathbf{Z}_qA)$ of $\mathbf{Z}_qA$. Then, we have a surjective map

$$\mathbf{Z}A_T \cap \mathcal{O} = R \to \mathbf{Z}_qA_{T'} \cap \mathcal{O}' = \mathbf{Z}_q[x, x^{-1}, y, y^{-1}, z]_{T_z'},$$

i.e. $R' = \mathbf{Z}_q[x, x^{-1}, y, y^{-1}, z]_{T_z'}$. It is worth noting that T_z is mapped into $\mathcal{O}'$. If $f \in \{f_1, \ldots, f_\nu\}$, $\phi \in \mathrm{Aut}(A)$, let

$$f^{\phi} = \sum_{i=m}^{n} a_i z^i, \qquad m < n, \quad a_i \in \mathbf{Z}[x, x^{-1}, y, y^{-1}], \quad a_m \neq 0, \quad a_n \neq 0.$$

Since q does not divide the integer coefficients of the a_i,

$$(z^{-m} f^{\phi})' = \sum_{i=m}^{n} a_i' z^{i-m}, \quad \text{where } a_m' \neq 0.$$

Thus, $(z^{-m} f^{\phi})'$ has z-value zero. We also note that the diagram

$$\begin{array}{ccc} \mathbf{Z}A_T & \to & \mathbf{Z}_qA_{T'} \\ \phi \downarrow & & \downarrow \phi \\ \mathbf{Z}A_T & \to & \mathbf{Z}_qA_{T'} \end{array}$$

commutes. If $\tau \in T$, then $\tau' \neq 0$, and moreover $T' \cap \mathcal{O}' = T_z'$. These remarks follow because for an irreducible element $\tau \in T$, $\tau \in T_z$ if and only if $\tau' \in T_z'$.

We "bar" an element or subset of $\mathcal{O}$ or $\mathcal{O}'$ to denote its image when we set $z = 0$.

The invertible elements of $\overline{R'}$ are products of the nonzero elements of $\mathbf{Z}_q$, x, x^{-1}, y, y^{-1} and irreducible factors of elements in $\overline{T_z'}$. We list below the irreducible elements (up to unit factor in $\mathbf{Z}_qA$) of T_z' and their images in $\overline{T_z'}$.

(i) $(1 - gz^k)$, where $gz^k = x^{\phi}$, $\phi \in \mathrm{Aut}(A)$, $g \in gp\langle x, y\rangle$, and $k > 0$.
$\overline{(1 - gz^k)} = 1$, a trivial unit.
(ii) $z^{-k}(1 - gz^k)$, where $gz^k = x^{\phi}$, $\phi \in \mathrm{Aut}(A)$, $g \in gp\langle x, y\rangle$, and $k < 0$.
$\overline{(z^{-k} - g)} = g$, again a trivial unit.
(iii) $(1 - g)$ where $g = x^{\phi}$, $\phi \in \mathrm{Aut}(A)$, and $g \in gp\langle x, y\rangle$.
$\overline{(1 - g)} = 1 - g$, an irreducible element in $\mathbf{Z}_qA$.

(iv) $z^{-m}f^{\phi} = \sum_{k=m}^{n} a_k z^{k-m}$, $f \in \{f_1', \ldots, f_\nu'\}$, $\phi \in \mathrm{Aut}(A)$, $m < n$, $a_k \in \mathbf{Z}_q[x, x^{-1}, y, y^{-1}]$, $a_m \neq 0$, $a_n \neq 0$.

$\overline{(z^{-m}f^{\phi})} = a_m$.

If $\alpha \in \mathbf{Z}_qA$, then supp α is the subset of A consisting of all elements whose coefficients in $\mathbf{Z}_q$ are nonzero when α is written as a linear combination of elements of A with coefficients from $\mathbf{Z}_q$. For $f^{\phi} = \sum_{k=m}^{n} a_k z^k$, as in (iv) above, notice that $\mathrm{supp}(a_k z^k) \cap \mathrm{supp}(a_j z^j) = \varnothing$ if $j \neq k$. Hence $f = \sum_{k=m}^{n} a_k^{\phi^{-1}}(z^{\phi^{-1}})^k$ where $\mathrm{supp}(a_k^{\phi^{-1}}(z^{\phi^{-1}})^k) \cap \mathrm{supp}(a_j^{\phi^{-1}}(z^{\phi^{-1}})^j) = \varnothing$ if $k \neq j$.

Let α be an irreducible factor of a_m. Then, $\alpha^{\phi^{-1}}$ is an irreducible factor of $a_m^{\phi^{-1}}$ or, equivalently, of $[a_m^{\phi^{-1}}(z^{\phi^{-1}})^m]$. Thus, $\alpha^{\phi^{-1}}$ is an irreducible factor of a partial sum of some $f \in \{f_1', \ldots, f_\nu'\}$ when f is written as a linear combination of elements from A with coefficients from $\mathbf{Z}_q$.

Let a be an element of $\mathbf{Z}$ such that a' does not have a pth root in $\mathbf{Z}_q$ for an odd prime p. Then, $x^{p^n} - a'$ is irreducible in $\mathbf{Z}_q[x]$ for all $n \geqslant 1$ [**5**, Theorem 51] and hence is irreducible in $\mathbf{Z}_q[x, x^{-1}, y, y^{-1}]$.

LEMMA 6. $\overline{(x^{p^n} - a')}$ *is not invertible in* $\overline{R'}$ *for infinitely many* n.

PROOF. It is clear that if $\overline{(x^{p^n} - a')}$ is invertible in $\overline{R'}$, then $\overline{(x^{p^n} - a')}$ comes from an element in $\overline{T_z'}$ of type (iv). Thus, there is $\Psi_n \in \mathrm{Aut}(A)$ such that $(x^{\psi_n^{-1}})^{p^n} - a'$ is an irreducible factor of a partial sum of some $f \in \{f_1', \ldots, f_\nu'\}$. There are only finitely many irreducible factors of partial sums of the f_i' up to unit factor in $\mathbf{Z}_qA$, whereas there are infinitely many irreducibles $(x^{\psi_n^{-1}})^{p^n} - a'$, up to unit factor in $\mathbf{Z}_qA$, as n varies. Thus, the lemma follows.

LEMMA 7. *Let* $\overline{\pi'} = \overline{(x^{p^n} - a')}$ *be a noninvertible element of* $\overline{R'}$ *where* $\pi = (x^{p^n} - a) \in \mathbf{Z}A$. *Then,* $\left(\begin{smallmatrix} 1 & \pi/q \\ 0 & 1 \end{smallmatrix}\right)$ *can be chosen as a double coset representative of* $(E_2(\mathbf{Z}A_T) \cap \mathrm{SL}_2(\mathcal{O}), \Gamma)$ *in* $\mathrm{SL}_2(\mathcal{O})$, *where* $\Gamma = \mathrm{SL}_2(\mathcal{O}) \cap \mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)}$.

PROOF. Suppose $\left(\begin{smallmatrix} 1 & \pi/q \\ 0 & 1 \end{smallmatrix}\right)$ cannot be so chosen. Then $\left(\begin{smallmatrix} 1 & \pi/q \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f & g \\ h & k \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & z\beta \\ \gamma & \delta \end{smallmatrix}\right)$ where $\left(\begin{smallmatrix} f & g \\ h & k \end{smallmatrix}\right) \in E_2(\mathbf{Z}A_T) \cap \mathrm{SL}_2(\mathcal{O})$ and $\left(\begin{smallmatrix} \alpha & z\beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$, i.e. $\alpha, \beta, \gamma, \delta$ are in $\mathcal{O}$.

$$\begin{pmatrix} 1 & \pi/q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta & z\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \delta - \pi\gamma/q & -z\beta + \pi\alpha/q \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} f & g \\ h & k \end{pmatrix}.$$

Thus, $f = \delta - \pi\gamma/q$, $g = \pi\alpha/q - z\beta$, $-\gamma = h$, $\alpha = k$. Since $z\beta q = \pi k - qg$ is contained in $R = \mathbf{Z}A_T \cap \mathcal{O}$, $\beta = \beta_1/q$ where $\beta_1 \in R$. Clearly, $\overline{\pi'}\,\overline{k'} = 0$, whence $\overline{k'} = 0$.

Next we compute $\overline{g'}$. By the above, $g = (\pi\alpha - z\beta_1)/q$, whence q divides $(\pi\alpha - z\beta_1)$ in R. $\alpha = \alpha_1 + z\alpha_2$ where $\alpha_1, \alpha_2 \in R$ and the numerator of α_1, which is an element of $\mathbf{Z}A$, remains unchanged when we set $z = 0$. Then,

$$g = [\pi\alpha_1 + z(\pi\alpha_2 - \beta_1)]/q,$$

and by setting $z = 0$, we see that q divides α_1 in R and therefore $(\pi\alpha_2 - \beta_1)$ as well. It is now evident that $\overline{g'} = \overline{\pi'}\,\overline{\alpha_3'}$ where $\alpha_1 = \alpha_3 q$. Thus,

$$\overline{\begin{pmatrix} f & g \\ h & k \end{pmatrix}'} = \begin{pmatrix} \overline{f'} & \overline{g'} \\ \overline{h'} & \overline{k'} \end{pmatrix} = \begin{pmatrix} \overline{f'} & \overline{\pi'}\,\overline{\alpha_1'} \\ \overline{h'} & 0 \end{pmatrix}$$

is an invertible matrix over $\overline{R'}$, whence $\overline{\pi'}$ is invertible over $\overline{R'}$, a contradiction. ■

LEMMA 8. *$E_2(\mathbf{Z}A_T) = U *_W V$ where $U = E_2(\mathbf{Z}A_T) \cap \mathrm{SL}_2(\mathcal{O})$, $V = E_2(\mathbf{Z}A_T) \cap \mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)}$, and $W = U \cap V$.*

PROOF. We observe that $E_2(\mathbf{Z}A_T)$ is generated by $\left(\begin{smallmatrix}z&0\\0&z^{-1}\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}1&0\\z^j/\tau&1\end{smallmatrix}\right)$ where $\tau \in T_z = T \cap \mathcal{O}$ and $j \geqslant 0$. Using Lemma 3 completes the argument. ■

COMPLETION OF PROOF OF THEOREM. From the Subgroup Theorem for Amalgamated Products [6], we see that since $E_2(\mathbf{Z}A_T)$ as an HNN group has no free part, $\left(\begin{smallmatrix}1&\pi/q\\0&1\end{smallmatrix}\right)$ must be a double coset representative of $(E_2(\mathbf{Z}A_T), \mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)})$ in $\mathrm{SL}_2(Q(\mathbf{Z}A))$. Thus, as a tree product, $\left(\begin{smallmatrix}1&\pi/q\\0&1\end{smallmatrix}\right)\mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)}\left(\begin{smallmatrix}1&-\pi/q\\0&1\end{smallmatrix}\right) \cap E_2(\mathbf{Z}A_T)$ is one of the vertices. But, from Lemma 8, $\left(\begin{smallmatrix}1&\pi/q\\0&1\end{smallmatrix}\right)\mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)}\left(\begin{smallmatrix}1&-\pi/q\\0&1\end{smallmatrix}\right) \cap E_2(\mathbf{Z}A_T)$ is contained in U. To complete the proof of the Theorem, we shall show that there is an element of $\left(\begin{smallmatrix}1&\pi/q\\0&1\end{smallmatrix}\right)\mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)}\left(\begin{smallmatrix}1&-\pi/q\\0&1\end{smallmatrix}\right) \cap \mathfrak{N}$ which is not in U. (Since our assumption that $\mathrm{Aut}(G)$ is a f.g. group implies that $\mathfrak{N} \subseteq E_2(\mathbf{Z}A_T)$, we will then have our desired contradiction.)

$$\left[1, \begin{pmatrix} 1 & 0 & 0 \\ \begin{array}{c}(x^{p^n}-a)^2(1-z)z^{-1}\\ -q(x^{p^n}-a)(1-x)(1-y)z^{-1}\end{array} & 1+q(x^{p^n}-a)(1-x)^2z^{-1} & -(x^{p^n}-a)^2(1-x)z^{-1} \\ \begin{array}{c}-q^2(1-x)^2(1-y)z^{-1}\\ +q(x^{p^n}-a)(1-x)(1-z)z^{-1}\end{array} & q^2(1-x)^3z^{-1} & 1-q(x^{p^n}-a)(1-x)^2z^{-1} \end{pmatrix}\right]$$

represents an element of $\mathrm{IA}(G)$ and maps into (see Lemma 2)

$$\begin{pmatrix} 1+q(x^{p^n}-a)(1-x)^2z^{-1} & -(x^{p^n}-a)^2(1-x)^2z^{-1} \\ q^2(1-x)^2z^{-1} & 1-q(x^{p^n}-a)(1-x)^2z^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & (x^{p^n}-a)/q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q^2(1-x)^2z^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -(x^{p^n}-a)/q \\ 0 & 1 \end{pmatrix}.$$

This last matrix is an element of $\left(\begin{smallmatrix}1&\pi/q\\0&1\end{smallmatrix}\right)\mathrm{SL}_2(\mathcal{O})^{\left(\begin{smallmatrix}1&0\\0&z\end{smallmatrix}\right)}\left(\begin{smallmatrix}1&\pi/q\\0&1\end{smallmatrix}\right) \cap \mathfrak{N}$ which is not in U and hence not in $E_2(\mathbf{Z}A_T)$. ■

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